

PRELIMINARY VERSION

SAMPLING AND INTERPOLATION IN DE BRANGES SPACES
WITH DOUBLING PHASE

JORDI MARZO, SHAHAF NITZAN AND JAN-FREDRIK OLSEN

ABSTRACT. The de Branges spaces of entire functions generalise the classical Paley-Wiener space of square summable bandlimited functions. Specifically, the square norm is computed on the real line with respect to weights given by the values of certain entire functions. For the Paley-Wiener space, this can be chosen to be an exponential function where the phase increases linearly. As our main result, we establish a natural geometric characterisation, in terms of densities, for real sampling and interpolating sequences in the case when the derivative of the phase function merely gives a doubling measure on the real line. Moreover, a consequence of this doubling condition, is that the spaces we consider are one component model spaces. A novelty of our work is the application to de Branges spaces of techniques developed by Marco, Massaneda and Ortega-Cerdà for Fock spaces satisfying a doubling condition analogue to ours.

1. INTRODUCTION

1.1. de Branges spaces and model spaces. An entire function E is a Hermite-Biehler function if it has no real zeroes and $|E^*(z)| < |E(z)|$ for $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, where $E^*(z) = \overline{E(\bar{z})}$. For such functions, the corresponding de Branges space is the Hilbert space of functions

$$H(E) = \{f \text{ entire} : f/E, f^*/E \in H^2(\mathbb{C}_+)\},$$

where $H^2(\mathbb{C}_+)$ is the Hardy space of the upper half-plane. The space $H(E)$ is equipped with the norm

$$\|f\|_{H(E)} := \sqrt{\int_{\mathbb{R}} \left| \frac{f(x)}{E(x)} \right|^2 dx}.$$

The main example of a de Branges space is the Paley-Wiener space, which is obtained by taking $E(z) = e^{-i\pi z}$. Inspired by this, it is usual to consider the polar decomposition

$$E(x) = |E(x)|e^{-i\phi(x)}.$$

The phase function ϕ is the increasing argument of E .

If $Z(E)$ is the zero set of E , then for some $\sigma \geq 0$ we have

$$\phi'(x) = \sigma + \sum_{z \in Z(E)} \operatorname{Im} \frac{1}{x - \bar{z}}.$$

This is easy to see, e.g. since E^*/E is an inner function on the upper half-plane with singular support only at infinity. Observe that when $\sigma = 0$, which is an assumption we will make from now on, the measure $\mu = \phi'(x)dx$ is a sum of harmonic measures with respect to the complex conjugate of the points in $Z(E)$.

We say that the de Branges space $H(E)$ satisfies the doubling condition if this measure is a doubling measure. I.e., if there exists a constant $C > 0$ such that $\mu(2I) \leq C\mu(I)$ for all intervals $I \subset \mathbb{R}$, where $2I$ is the interval co-centric to I of double length.

The measure μ induces a natural metric on \mathbb{R} , namely

$$d_\phi(x, y) = |\phi(x) - \phi(y)|.$$

To motivate the significance of this metric, we consider the reproducing kernels of the space $H(E)$. These are functions $K_w \in H(E)$, for $w \in \mathbb{C}$, such that $\langle f, K_w \rangle = f(w)$ for every $f \in H(E)$. It is not hard to see that in the inner product implied by the norm of $H(E)$, we have

$$K_w(z) = K(z, w) = \frac{i}{2\pi} \frac{E(z)\overline{E(w)} - E^*(z)\overline{E^*(w)}}{(z - \bar{w})}, \quad z, w \in \mathbb{C}.$$

We denote by $\tilde{K}_w = K_w/\|K_w\|$ the normalised reproducing kernel. From this, it follows that for $x, y \in \mathbb{R}$ we have

$$K_y(x) = \frac{|E(x)E(y)|}{\pi} \frac{\sin(\phi(x) - \phi(y))}{(x - y)}, \quad \tilde{K}_y(x) = \frac{|E(x)| \sin(\phi(x) - \phi(y))}{\sqrt{\pi\phi'(y)}(x - y)}.$$

So, if we, for some fixed $\alpha \in [0, 2\pi)$, let $\{\omega_n\}$ be a sequence of points such that $\phi(\omega_n) = \alpha + \pi n$, then the normalised reproducing kernels $\{\tilde{K}_{\omega_n}\}$ form an orthonormal sequence. It is a theorem of de Branges [13, p. 55], that for every such α , except at most one, this sequence of reproducing kernels is an orthonormal basis for $H(E)$.

The model spaces K_Θ^2 , where Θ is an inner function, is the subspace of the classical Hardy space $H^2(\mathbb{C}_+)$ defined by

$$K_\Theta^2 = H^2(\mathbb{C}_+) \cap (\Theta H^2(\mathbb{C}_+))^\perp.$$

It is a simple observation that $f \mapsto f/E$ is a unitary homeomorphism between $H(E)$ and the model subspace K_Θ^2 , where $\Theta = E^*/E$ is the inner function. In this way, it is possible to show that the de Branges spaces correspond exactly to the model subspaces with meromorphic inner functions [15].

A model space K_Θ^2 satisfies the one component condition if there exists an $\epsilon \in (0, 1)$ such that the level set

$$\{z : |\Theta(z)| < \epsilon\}$$

is connected. These spaces were introduced by Cohn [11, 12]. By a characterization due to Aleksandrov [2] and Lemma 2 below, the meromorphic model spaces satisfying the one component condition essentially correspond to the class of de Branges spaces satisfying a local doubling condition (Corollary 1).

1.2. The Fock space approach to sampling and interpolation. Sampling and interpolating sequences can be seen as dual concepts. We give the definitions for a general Hilbert space H of functions, analytic on a domain $\Omega \subset \mathbb{C}$, with reproducing kernels k_z .

Definition 1. Let $\Gamma \subset \Omega$ be a sequence of points. Then:

- (a) Γ is called sampling for H if there exists constants $A, B > 0$ such that for all $f \in H$ we have

$$A\|f\|^2 \leq \sum_{\gamma \in \Gamma} |f(\gamma)|^2 / \|k_\gamma\|^2 \leq B\|f\|^2.$$

(If only the upper inequality holds, Γ is said to be a Bessel sequence.)

- (b) Γ is called interpolating for H if, for every sequence $\{a_\gamma\}$ satisfying

$$\sum_{\gamma \in \Gamma} |a_\gamma|^2 / \|k_\gamma\|^2 < \infty,$$

there exist $f \in H$ such that $f(\gamma) = a_\gamma$.

Observe that a sequence Γ is sampling if and only if the normalized reproducing kernels $\{k_\gamma / \|k_\gamma\|\}$ form a frame in H , while Γ is interpolating if and only if $\{k_\gamma / \|k_\gamma\|\}$ is a Riesz sequence in H (for these, and related notions, see [28]).

We use the following notion of separation.

Definition 2. A real sequence Γ is called ϕ -separated when there exists an $\epsilon > 0$ such that $d_\phi(\gamma, \gamma') > \epsilon$ whenever $\gamma, \gamma' \in \Gamma$ and $\gamma \neq \gamma'$.

Observe that the points ω_n for which $\phi(\omega_n) = \alpha + \pi n$ are at an integer distance from each other with respect to this metric.

To study the *real* sampling and interpolating sequences for $H(E)$, we introduce the following Beurling-type densities. Recall that $\mu(I) = \int_I \phi'(x) dx$.

Definition 3. Given a ϕ -separated sequence $\Gamma = \{\gamma_k\} \subset \mathbb{R}$, and a measure μ as above, the lower and upper uniform densities of Γ are given by

$$D_\phi^-(\Gamma) = \liminf_{r \rightarrow \infty} \inf_{I: \mu(I)=r} \frac{|\Gamma \cap I|}{r},$$

and

$$D_\phi^+(\Gamma) = \limsup_{r \rightarrow \infty} \sup_{I: \mu(I)=r} \frac{|\Gamma \cap I|}{r},$$

where the I are bounded intervals in \mathbb{R} and $|\Gamma \cap I|$ denotes the number of points in $\Gamma \cap I$.

Our approach is motivated by work on sampling and interpolating sequences in the so-called weighted Paley-Wiener spaces [17], and in the Fock space setting [19]. I.e., the space

$$\mathcal{F} = \left\{ f \text{ entire} : \int_{\mathbb{C}} |f(z)|^2 e^{-\Phi(z)} dm(z) < \infty \right\},$$

where dm is the planar Lebesgue measure and Φ is a subharmonic function on \mathbb{C} .

In [26, 29], the sampling and interpolating sequences are characterized in terms of Beurling densities for the classical Fock space $\Phi(z) = |z|^2$ (for which $\Delta\Phi = 1$). In subsequent work, the L^p case for spaces with $\Delta\Phi \simeq 1$ was studied [24]. Finally, in [19], the sampling and interpolating sequences are characterized in terms of Beurling-type densities for spaces where $\Delta\Phi$ gives a doubling measure on \mathbb{C} .

The analogy to the de Branges setting becomes clearer, perhaps, when we observe that the function

$$\Phi(z) := \begin{cases} \log |E(z)| & \operatorname{Re} z \geq 0, \\ \log |E^*(z)| & \operatorname{Re} z < 0, \end{cases}$$

has a Laplacian which is supported on \mathbb{R} and satisfies $\Delta\Phi = 2\phi'$ there (see Lemma 4). With this notation, the classical case (Paley-Wiener space) corresponds to $\Delta\Phi = 1$ when restricted to \mathbb{R} . Since functions in $H(E)$ have similar local properties as those in the corresponding Fock space, we are able to use Fock space techniques in our setting.

Recall that the density results for the Paley-Wiener case due to Beurling, Jaffard and Seip [7, 16, 26] extend essentially without changes to the so called Weighted Paley-Wiener spaces, [17]. In particular, they show that any such space is equal to a de Branges space for which the zeroes of the Hermite-Biehler function do not approach the real axis, and for which $\phi'(x) \simeq 1$. The family of spaces we study contains spaces which were not studied in that setting. However, the converse is also true, they give an example of a de Branges space that is equal to the classical Paley-Wiener space, but for which the derivative of the phase of the Hermite-Biehler function is not doubling.

In an example where the phase function is made to oscillate by placing zeroes exponentially sparse on a horizontal line, Baranov [4] proved that, in contrast to what is known for Paley-Wiener spaces, there are separated sequences that are not Bessel sequences. This indicates that density results like the ones of theorems 1 and 2, cannot hold in this non-doubling situation. Recently, this was studied in more generality by Belov, Mengestie and Seip [6], who give a complete characterisation of Bessel sequences for a class of “small” de Branges spaces. (See also the discussion following the statement of Theorem 2.)

Having said this, it appears natural to study the de Branges spaces with doubling measure $\phi'(x)dx$ as an extension of Paley-Wiener spaces, using techniques adapted from the Fock space setting.

Observe that this approach suggests the study of the de Branges analogs to the “large” and “small” Fock spaces introduced in [8] and [9], among other problems. Some of this work has been already done by Mengestie in his forthcoming doctoral dissertation [21] (see also [6]).

1.3. Main result. We obtain the following description of sampling and interpolating sequences for the de Branges spaces satisfying the doubling condition.

Theorem 1. *Let E be a Hermite-Biehler function with phase function ϕ , and suppose that $\phi'(x)dx$ is a doubling measure on \mathbb{R} .*

If Γ is a real sampling sequence for $H(E)$, then it is a finite union of ϕ -separated subsequences and there exists a ϕ -separated subset $\Gamma' \subset \Gamma$ which is sampling with $D_\phi^-(\Gamma') \geq 1/\pi$. Conversely, if a real sequence Γ is a finite union of ϕ -separated sequences and there is a ϕ -separated subsequence $\Gamma' \subset \Gamma$ such that $D_\phi^-(\Gamma') > 1/\pi$, then Γ is sampling.

The interpolation part reads as follows.

Theorem 2. *Let E be a Hermite-Biehler function with phase function ϕ , and suppose that $\phi'(x)dx$ is a doubling measure on \mathbb{R} .*

If Γ is a real interpolating sequence for $H(E)$, then it is ϕ -separated with $D_\phi^+(\Gamma) \leq 1/\pi$. Conversely, if a real sequence Γ is ϕ -separated and $D_\phi^+(\Gamma) < 1/\pi$, then Γ is interpolating.

As we pointed out above, for some spaces $H(E)$, where the derivative of the phase function ϕ is highly oscillating, such results are no longer valid [4, 6]. An important motivation to study such de Branges spaces, was to try to find a counterexample to Feichtinger's conjecture, which in our setting says that a sampling sequence can be written as a finite union of interpolating sequences.

However, it has been recently proved that Feichtinger's conjecture for reproducing kernels holds true both in this case [6], and in the case of one-component model subspaces [5]. (Observe that for de Branges spaces with doubling phase, this also follows from theorems 1 and 2.) This seems to suggest that counterexamples, if there are any, should be found when the phase goes from doubling to highly oscillatory.

1.4. Outline of the paper. In section 2, we prove some preliminary results about sampling and interpolating sequences in de Branges spaces with doubling and locally doubling measures. In section 3, following Beurling we construct a multiplier function. In section 4, we first prove the necessity of the density conditions in theorems 1 and 2 using the techniques developed by Ramanathan and Steger. Then, we prove the sufficiency for interpolation using a Lagrange interpolating function and the sufficiency for sampling following the weak limit techniques developed by Beurling.

2. PRELIMINARIES

We begin by establishing some preliminary results needed in the rest of the paper, some of which are previously known. We only provide full proofs if the result is new, or there is some novelty in the argument.

First some notation. For $x \in \mathbb{R}$ and $r > 0$, we set $I_\phi(x, r) = \{y \in \mathbb{R} : |\phi(x) - \phi(y)| < r\}$, and let $D_\phi(x, r)$ denote the disk in \mathbb{C} with both diameter and intersection with \mathbb{R} equal to $I_\phi(x, r)$. For $\phi(x) = x$, we simply write $I(x, r)$

and $D(x, r)$, respectively. By dm , we denote the Lebesgue measure on \mathbb{C} , and we use the symbol $f \lesssim g$ to denote that there exists a constant $C > 0$ such that $f \leq Cg$ for real valued functions f and g . The symbols \gtrsim and \simeq are defined analogously. For an entire function f , we denote its zero set by $Z(f)$.

2.1. Locally doubling measures and phase functions. Recall from the introduction that a positive Borel measure μ is doubling if there exists a constant $C > 0$ such that $\mu(2I) \leq C\mu(I)$ for every bounded interval I . We say that μ is locally doubling if the same holds true whenever $\mu(2I) < 1$.

(Observe that this notion of locally doubling measures is different than the one defined with the same name in [22].)

We state a standard result for doubling measures, see [20, Lemma 2.1.] or [22, Lemma 1].

Lemma 1. *Let μ be a positive (Borel) doubling measure. Then there exists $0 < \gamma$ such that for any pair of intervals I, I' with radius r, r' such that $r > r'$, $I \cap I' \neq \emptyset$*

$$\left(\frac{\mu(I)}{\mu(I')} \right)^\gamma \lesssim \frac{r}{r'} \lesssim \left(\frac{\mu(I)}{\mu(I')} \right)^{1/\gamma} \quad (1)$$

Remark 1. A version also holds for locally doubling measures. Indeed, one gets the same conclusion (1) supposing that $\mu(I \cup I') \leq 1$.

For meromorphic inner functions Θ , it was pointed out to us by A. Baranov that there is a close connection between the one component condition and the doubling condition on the phase. In the following lemma, we essentially prove that for such functions, the local doubling condition on the phase is equivalent to the one component condition (Corollary 1).

Lemma 2. *Let E be a Hermite-Biehler function with phase ϕ . Then the following are equivalent.*

- (a) *The measure $\phi'(x)dx$ is locally doubling.*
- (b) *There exist constants such that $\phi'(x) \simeq \phi'(y)$ whenever $|\phi(x) - \phi(y)| \leq 1$.*
- (c) *The inequality $|\phi''(x)| \leq C(\phi'(x))^2$ holds.*

Proof. a) \iff b): Suppose that μ is locally doubling, and let $C > 0$ be the local doubling constant of μ for intervals I such that $\mu(2I) \leq \pi/2$. We first show that for x, y in any interval I of μ -measure $\pi/2C$, then $\phi'(x) \simeq \phi'(y)$.

Observe that if J is an interval, co-centric with I , and such that $\mu(J) = \pi/2$ then by local doubling $\mu(J/2) \geq \mu(J)/C = \pi/2C$. This implies that $I \subset J/2$, and so $2I \subset J$. This implies $\mu(2I) \leq \pi/2$.

Let D denote the disk in \mathbb{C} such that $D \cap \mathbb{R} = 2I$. Since $\phi'(x)$ is a sum of Poisson kernels P_z with respect to the zeros of E^* , it follows that no such zero belongs to D . Indeed, if this was the case then the harmonic measure of $2I$ with respect to this point, which is smaller than $\mu(2I)$, would exceed $\pi/2$.

For each zero z of E^* , it now follows easily that $P_z(x) \simeq P_z(y)$, which yields $\phi'(x) \simeq \phi'(y)$.

By a covering argument, we obtain (b). The converse is immediate.

b) \iff c): If we denote the zeroes of E^* by $z_n = a_n + ib_n$, then by the triangle inequality,

$$|\phi''(x)| \leq 2 \sum_n \frac{b_n |x - a_n|}{((x - a_n)^2 + b_n^2)^2} := 2(A + B),$$

where we have split the sum according to whether $|x - a_n| \leq b_n$ or $|x - a_n| > b_n$, respectively. The first estimate is clear,

$$A \leq \sum_{|x - a_n| \leq b_n} \frac{b_n^2}{((x - a_n)^2 + b_n^2)^2} \leq \left(\sum_n \frac{b_n}{(x - a_n)^2 + b_n^2} \right)^2 = (\phi'(x))^2.$$

To obtain the second estimate, we observe that for this case $1 \lesssim \phi'(x)|x - a_n|$. Indeed, first suppose that $|\phi(x) - \phi(a_n)| > 1$. Letting u be the point between a_n and x such that $|\phi(x) - \phi(u)| = 1$, it readily follows that

$$1 = |\phi(x) - \phi(u)| \simeq \phi'(x)|x - u| \leq \phi'(x)|x - a_n|$$

On the other hand, if $|\phi(x) - \phi(a_n)| \leq 1$, then we know that $\phi'(a_n) \sim \phi'(x)$. Since $1 \leq b_n \phi'(a_n)$ (always!), the condition $b_n < |x - a_n|$ gives the required estimate.

It now follows that

$$B \leq \sum_{|x - a_n| > b_n} \frac{b_n}{(x - a_n)^2 + b_n^2} \cdot \frac{1}{|x - a_n|} \lesssim \phi'(x) \sum_n \frac{b_n}{(x - a_n)^2 + b_n^2} = (\phi'(x))^2.$$

The converse implication holds in general for any increasing function ϕ . Indeed, let $d_\phi(x, y) < 1$ and suppose that $\phi'(x) < \phi'(y)$, then

$$\log \frac{\phi'(y)}{\phi'(x)} = \left| \log \frac{\phi'(y)}{\phi'(x)} \right| = \left| \int_x^y \frac{\phi''(t)}{\phi'(t)} dt \right| \leq C \left| \int_x^y \phi'(t) dt \right| \leq C.$$

□

For inner functions Θ on the unit disk \mathbb{D} with phase ϕ , it is proved by Aleksandrov [2] that condition (c) above, under some mild additional restrictions that are automatically satisfied when Θ is a meromorphic Blaschke product, is equivalent to Θ being one component. Following the argument of Aleksandrov, it is clear that the same result holds on the upper half-plane as long as one makes the additional hypothesis that $\Theta = E^*/E$ has infinity in its spectrum (cf. [1, Theorem 3.4]). Hence, we obtain the following consequence.

Corollary 1. *Let E be a Hermite-Biehler function with phase ϕ , and suppose that a subsequence of zeroes converges to infinity. Then $\phi'(x)dx$ gives a locally doubling measure if and only if the inner function E^*/E satisfies the one component condition.*

We point out that there exists a Hermite-Biehler function E where the phase ϕ satisfies the local doubling condition, but not the doubling condition. This example is obtained by letting E be the entire function with zeroes $z_n = -i2^n$ for $n \in \mathbb{N}$. Indeed, then $\phi'(x) \simeq (1+x)^{-1}$.

Rewriting parts of the two previous lemmas in the notation of the metric $d_\phi(x, y) = |\phi(x) - \phi(y)|$ we easily obtain the following. It is analogue to [19, Lemma 4].

Lemma 3. *Let E be a Hermite-Biehler function with phase ϕ . Then there are constants depending on $r > 0$ such that the following hold:*

- a) *If the measure $\phi'(x)dx$ is locally doubling and $d_\phi(x, y) \leq r$, then $d_\phi(x, y) \simeq \phi'(x)|x - y|$.*
- b) *If the measure $\phi'(x)dx$ is doubling and $d_\phi(x, y) > r$, then $(\phi'(x)|x - y|)^{1/\alpha} \lesssim d_\phi(x, y) \lesssim (\phi'(x)|x - y|)^\alpha$, for some $\alpha > 0$ depending only on the doubling constant of the measure.*

Remark 2. When $\phi'(x)dx$ is a doubling measure, it follows by Lemma 3 that there exists some $\delta > 0$, such that for $r > 0$ and $d_\phi(x, y) > r$, then

$$\phi'(y) \lesssim d_\phi(x, y)^\delta \phi'(x),$$

with the constant depending on r .

Remark 3. Observe that Lemma 3 still holds if we merely assume ϕ to be a smooth increasing function satisfying condition (b) of Lemma 2.

2.2. Pointwise estimates of functions in de Branges spaces. The following lemma, while simple, explains the analogy between de Branges spaces with a doubling phase function and Fock spaces with a doubling laplacian.

Lemma 4. *Let E be a Hermite-Biehler function, and set*

$$\Phi(z) := \begin{cases} \log |E(z)| & \text{if } \operatorname{Im} z \geq 0, \\ \log |E^*(z)| & \text{if } \operatorname{Im} z < 0. \end{cases} \quad (2)$$

Then

$$\Delta \Phi(z) = \begin{cases} 2\phi'(z) & z \in \mathbb{R} \\ 0 & z \notin \mathbb{R}. \end{cases}$$

Proof. The function $\log E$ is analytic in a neighborhood of $x \in \mathbb{R}$, so by the Cauchy-Riemann equations, $\partial_y(\log |E|)(x) = -\partial_x(\arg E)(x) = \phi'(x)$. The result now follows by an argument using Green's formula. \square

We now show that for functions in a de Branges space satisfying the local doubling property, a subharmonic-type estimate holds. This is a crucial tool, which in the context of Fock spaces corresponds to [19, Lemma 19].

Lemma 5. *Let E be a Hermite-Biehler function with phase function ϕ , and let Φ be given by (2). If $\phi'(x)dx$ is a locally doubling measure on \mathbb{R} , then there are constants only depending on $r > 0$ such that for all $f \in H(E)$ and $x \in \mathbb{R}$ we have*

$$\left| \frac{f(x)}{E(x)} \right|^2 \lesssim \phi'(x)^2 \int_{D_\phi(x,r)} |f(\xi)|^2 e^{-2\Phi(\xi)} dm(\xi), \quad (3)$$

and

$$\left| \left(\frac{f}{E} \right)'(x) \right|^2 \lesssim \phi'(x)^4 \int_{D_\phi(x,r)} |f(\xi)|^2 e^{-2\Phi(\xi)} dm(\xi). \quad (4)$$

Proof. We first observe that an inequality of the type (3) follows immediately by subharmonicity for a Euclidean disk centered at $z \in \mathbb{C}$, with radius $1/\phi'(\operatorname{Re} z)$, and which does not intersect \mathbb{R} . We now prove that such an inequality holds even if the disk intersects \mathbb{R} , and deduce the lemma from it.

Given $\rho > 0$, fix $z \in \mathbb{C}$ and set $\operatorname{Re} z = x$. Suppose that the euclidean disc $D(z, \rho/\phi'(x))$ intersects \mathbb{R} . By Green's formula, there exists a harmonic function h on $D(z, 2\rho/\phi'(x))$ such that

$$h(\xi) = \Phi(\xi) - \Phi(z) + \underbrace{\int_{D(z, 2\rho/\phi'(x))} (G(\eta, \xi) - G(\eta, z)) \Delta \Phi(\eta) dm(\eta)}_{=: B(\xi)}, \quad (5)$$

where $G(\eta, \xi)$ is Green's function for $D(z, 2\rho/\phi'(x))$ with pole at ξ . Observe that $h(z) = 0$.

We proceed to estimate $|B(\xi)|$ for $\xi \in D(z, \rho/\phi'(x))$. Set $J = D(z, 2\rho/\phi'(x)) \cap \mathbb{R}$. By Lemma 4 we have $\Delta \Phi(\eta) = 2\phi'(\eta)$, and so

$$B(\xi) = 2 \int_J (G(\eta, \xi) - G(\eta, z)) \phi'(\eta) d\eta.$$

It is clear that $J \subset I(x, 2\rho/\phi'(x))$, so by Lemma 2 and the explicit form of the Green function, we have

$$\int_J G(\eta, \xi) \phi'(\eta) d\eta \simeq \phi'(x) \int_J G(\eta, \xi) d\eta \leq C\rho.$$

Applying this estimate to both $G(\eta, z)$ and $G(\eta, \xi)$, we obtain $|B(\xi)| \leq C\rho$.

Let $H = h + i\tilde{h}$ be an analytic completion of h . Since H is purely imaginary at the point z , we can use the estimate on B , and the subharmonicity of $|e^{-H(z)} f(z)|^2$, to obtain

$$|f(z)e^{-\Phi(z)}|^2 = |e^{-H(z)} f(z)|^2 e^{-2\Phi(z)} \lesssim \phi'(x)^2 \int_{D(z, \rho/\phi'(x))} |f(\xi)|^2 e^{-2\Phi(\xi)} dm(\xi), \quad (6)$$

where the implicit constant only depends on ρ .

The inequality (3) now follows by observing that there exists an $\rho > 0$, depending only on r , such that the ball $D(x, 2\rho/\phi'(x))$ is contained in $D_\phi(x, r)$.

For (4), we use a simple contour argument involving the Cauchy integral formula, and take the derivative to see that for any $\epsilon > 0$

$$\left(\frac{f}{E}\right)'(x) = \frac{1}{2\pi i} \int_{|z-x|=\epsilon/\phi'(x)} \frac{f(z)e^{-\Phi(z)}}{(z-x)^2} dz.$$

The conclusion now follows by choosing $\epsilon > 0$ small enough, using the triangle inequality and applying (6). \square

2.3. Carleson measures and a Bernstein inequality. In this, and the next subsection, we use the Fock space point of view to obtain the basic tools needed to study sampling and interpolating sequences for de Branges spaces satisfying the doubling condition. Most of these results have been proved by different methods in the context of one component model spaces.

Recall that a positive Borel measure ν on \mathbb{C}_+ is called Carleson for the space $H^2(\mathbb{C}_+)$ if there exists a constant such that for all $f \in H^2(\mathbb{C}_+)$

$$\int_{\mathbb{C}_+} |f(z)|^2 d\nu(z) \lesssim \|f\|_{H^2(\mathbb{C}_+)}^2.$$

A well-known characterization, due to Carleson [10] and Shapiro and Shields [30], says that a positive Borel measure ν is a Carleson measure for $H^2(\mathbb{C}_+)$ if and only if $\nu(C_I) \lesssim |I|$ for all bounded intervals $I \subset \mathbb{R}$, where C_I is the square in \mathbb{C}_+ with base I . It is clear that the above holds true, with obvious modifications, for the space $H^2(\mathbb{C}_-)$.

Carleson measures for model spaces were first considered by Cohn [11, 12]. For positive measures supported on \mathbb{R} this notion carries over directly to de Branges spaces: we say that a measure ν is a Carleson measure for $H(E)$ if

$$\int_{\mathbb{R}} \frac{|f(x)|^2}{|E(x)|^2} d\nu(x) \lesssim \|f\|_{H(E)}^2, \quad f \in H(E).$$

We begin with a simple lemma.

Lemma 6. *Let E be a Hermite-Biehler function with phase function ϕ , and set*

$$D = \bigcup_{t \in \mathbb{R}} D_\phi(t, 1).$$

If $\phi'(x)dx$ is a locally doubling measure on \mathbb{R} , then the measure

$$d\nu^\pm(z) = \phi'(\operatorname{Re} z) \chi_{D \cap \mathbb{C}_\pm}(z) dm(z)$$

is Carleson for $H^2(\mathbb{C}_\pm)$.

Proof. Let $I \subset \mathbb{R}$ be an interval with center $t \in \mathbb{R}$, and denote by C_I the Carleson square with base I . First, suppose that $\mu(I) \leq 1$. By Lemma 2, we have $\phi'(t) \simeq \phi'(s)$ for all $s \in I$, and furthermore $|I|\phi'(t) \lesssim 1$. Hence,

$$\nu^+(C_I) \leq \int_{C_I} \phi'(\operatorname{Re} z) dm(z) \lesssim \phi'(t)|I|^2 \lesssim |I|.$$

Next, suppose that $\mu(I) > 1$. Let $\{I_j\}_{j=1}^N$ be a cover of I by intervals with disjoint interiors such that $\mu(I_j) = 1$. It follows from the doubling condition that

$$|I| \simeq \sum_{j=1}^N |I_j|,$$

with a uniform control of the implicit constants and N .

Let \widetilde{C}_{I_j} denote the rectangle with base I_j and height $h_j = \sup_{s \in I_j} 1/\phi'(s)$. Since $\phi'(t) \simeq \phi'(s)$ for $s, t \in I_j$, it now follows that

$$\nu^+(C_I) \leq \sum_{j=1}^N \int_{\widetilde{C}_{I_j}} \phi'(\operatorname{Re} z) dm(z) \simeq \sum_{j=1}^N |I_j| \simeq |I|.$$

For ν^- a similar argument works. □

Remark 4. A discrete version of this result also holds i.e. if $\{t_n\}$ is a ϕ -separated sequence of points in \mathbb{R} . Then $d\nu^\pm(z) = \sum_{n \in \mathbb{Z}} \phi'(t_n) \chi_{D_\phi(t_n, 1) \cap \mathbb{C}_\pm}(z) dm(z)$ is a Carleson measure on $H^2(\mathbb{C}_\pm)$.

The following Bernstein-type inequality is a special case of [3, Corollary 1.5]. We include a simple proof using our techniques.

Lemma 7. *Let E be a Hermite-Biehler function with phase function ϕ . If $\phi'(x)dx$ is a locally doubling measure on \mathbb{R} , then there exists a constant $C > 0$ such that*

$$\left\| \frac{(f/E)'}{\phi'} \right\|_{L^2(\mathbb{R})} \leq C \|f\|_{H(E)}, \quad f \in H(E).$$

Proof. Let $D = \bigcup_{t \in \mathbb{R}} D_\phi(t, 1)$. Observe that $z \in D_\phi(t, 1)$ implies $t \in I_\phi(\operatorname{Re} z, 1)$. Lemma 5, followed by Fubini's theorem yields

$$\begin{aligned} \left\| \frac{(f/E)'}{\phi'} \right\|_{L^2(\mathbb{R})}^2 &\lesssim \int_{\mathbb{R}} \phi'(t)^2 \int_{D_\phi(t, 1)} |f(z)|^2 e^{-2\Phi(z)} dm(z) dt \\ &\lesssim \int_D |f(z)|^2 e^{-2\Phi(z)} \left(\int_{I(\operatorname{Re} z, c)} \phi'(t) dt \right) \phi'(\operatorname{Re} z) dm(z) \\ &= \int_D |f(z)|^2 e^{-2\Phi(z)} \phi'(\operatorname{Re} z) dm(z). \end{aligned}$$

The conclusion now follows by Lemma 6. □

Remark 5. The above lemma also holds for any $1 \leq p < \infty$ using the obvious generalizations.

A well-known result by Volberg and Treil [31] in the context of one component model spaces implies the following characterization.

Lemma 8. *Let E be a Hermite-Biehler function with phase function ϕ . If $\phi'(x)dx$ is a locally doubling measure on \mathbb{R} , then a positive Borel measure ν with support in \mathbb{R} is a Carleson measure for the space $H(E)$ if and only if*

$$\int_I \phi'(x) d\nu(x) \lesssim 1 \quad (7)$$

for all intervals $I \subset \mathbb{R}$ with $\int_I \phi'(x) dx = 1$.

2.4. Basic results about sampling and interpolating sequences. The following result is an immediate consequence of Lemma 8.

Corollary 2. *Let E be a Hermite-Biehler function with phase function ϕ . If $\phi'(x)dx$ is a locally doubling measure on \mathbb{R} , then a real sequence Γ is Bessel if and only if it is a finite union of uniformly ϕ -separated sequences.*

It follows that we may always assume that sampling sequences are ϕ -separated, which is a fact we need to prove our density results. We now establish some other basic results.

The next lemma is a standard tool to prove perturbative results. Indeed, combined with standard arguments, stability results for Bessel sequences, sampling sequences and interpolating sequences follow (see e.g. [32, chapter 4]).

While we include a simple proof of the lemma to illustrate the use of our techniques, we do not work out all of these consequences, instead we refer the interested reader to Baranov [4], where such results are obtained for general one component model spaces using other tools.

Lemma 9. *Let E be a Hermite-Biehler function with phase function ϕ , and suppose that $\{\lambda_n\}$ is a real ϕ -separated sequence. If $\phi'(x)dx$ is a locally doubling measure on \mathbb{R} , then for every $\delta > 0$ and real sequence $\{\gamma_n\}$ that satisfies $\sup_n d_\phi(\lambda_n, \gamma_n) < \delta$, there exists a constant $C(\delta) > 0$ such that*

$$\sum_n |\langle f, \tilde{K}_{\lambda_n} - \tilde{K}_{\gamma_n} \rangle|^2 < C(\delta) \|f\|_{H(E)}^2, \quad \forall f \in H(E).$$

Moreover, $C(\delta)$ can be chosen to satisfy $C(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Proof. We begin by estimating

$$\begin{aligned} |\langle f, \tilde{K}_{\lambda_n} - \tilde{K}_{\gamma_n} \rangle|^2 &= \left| \frac{f(\lambda_n)}{E(\lambda_n)\sqrt{\phi'(\lambda_n)}} - \frac{f(\gamma_n)}{E(\gamma_n)\sqrt{\phi'(\gamma_n)}} \right|^2 \\ &= \left| \int_{\gamma_n}^{\lambda_n} \left(\frac{f}{E\sqrt{\phi'}} \right)'(x) dx \right|^2 \end{aligned}$$

$$\leq 2 \underbrace{\left| \int_{\gamma_n}^{\lambda_n} \left(\frac{f}{E} \right)'(x) \frac{1}{\sqrt{\phi'(x)}} dx \right|^2}_{(*)} + \underbrace{\left| \int_{\gamma_n}^{\lambda_n} \frac{f(x)}{E(x)} \frac{\phi''(x)}{(\sqrt{\phi'(x)})^3} dx \right|^2}_{(**)}.$$

To estimate $(*)$ we apply the Cauchy-Schwartz inequality to obtain

$$(*) \leq \left(\int_{\gamma_n}^{\lambda_n} \phi'(x) dx \right) \left(\int_{\gamma_n}^{\lambda_n} \left| \frac{(f/E)'(x)}{\phi'(x)} \right|^2 dx \right) \leq \delta \int_{\gamma_n}^{\lambda_n} \left| \frac{(f/E)'(x)}{\phi'(x)} \right|^2 dx.$$

In the case of $(**)$, the Cauchy-Schwartz inequality followed by Lemma 2 yields

$$(**) \leq \left(\int_{\gamma_n}^{\lambda_n} \phi'(x) dx \right) \left(\int_{\gamma_n}^{\lambda_n} \left| \frac{f(x)}{E(x)} \right|^2 \left(\frac{\phi''(x)}{(\phi'(x))^2} \right)^2 dx \right) \lesssim \delta \int_{\gamma_n}^{\lambda_n} \left| \frac{f(x)}{E(x)} \right|^2 dx.$$

Let I_n denote interval with end points λ_n, γ_n . For $\delta > 0$ small enough, the intervals I_n do not overlap, and we simply sum the previous two estimates up to get

$$\sum_n |\langle f, \tilde{K}_{\lambda_n} - \tilde{K}_{\gamma_n} \rangle|^2 \lesssim \delta \left(\int_{\mathbb{R}} \left| \frac{(f/E)'(x)}{\phi'(x)} \right|^2 dx + \int_{\mathbb{R}} \left| \frac{f(x)}{E(x)} \right|^2 dx \right).$$

The desired estimate for $\delta > 0$ small, now follows from Lemma 7. To see that we get an inequality for any $\delta > 0$, with some constant, it is enough to note that in general there is a uniform bound, only depending on $\delta > 0$ and E , for how many times the intervals I_n overlap. \square

We end this section on preliminaries with two standard results on separation that we need in the proof of theorems 1 and 2. They do not seem to have appeared previously in our context. For a proof in the Paley-Wiener case see [27, Lemma 3.11, 3.12]. We provide a proof of the first lemma to illustrate again the use of our techniques.

Lemma 10. *Let E be a Hermite-Biehler function with phase function ϕ . If $\phi'(x)dx$ is a locally doubling measure on \mathbb{R} , then every interpolating sequence Γ is ϕ -separated.*

Proof. For $\gamma \in \Gamma$ there exists a function such that for $\gamma' \in \Gamma$ we have

$$f_\gamma(\gamma') = \begin{cases} 0 & \gamma' \neq \gamma, \\ E(\gamma) \sqrt{\phi'(\gamma)} & \gamma' = \gamma, \end{cases}$$

with $\|f_\gamma\|_{H(E)} \lesssim 1$. For $\gamma \neq \gamma'$, this implies

$$\sqrt{\phi'(\gamma)} = \left| \frac{f_\gamma(\gamma)}{E(\gamma)} - \frac{f_\gamma(\gamma')}{E(\gamma')} \right| = |\gamma - \gamma'| \left| \left(\frac{f_\gamma}{E} \right)'(x) \right|,$$

for some x in the interval with end points γ, γ' . By Lemma 5, this is less than

$$|\gamma - \gamma'| \phi'(x)^{3/2} \underbrace{\left(\phi'(x) \int_{D_\phi(x,1)} |f_\gamma(\xi)|^2 e^{-2\Phi(\xi)} dm(\xi) \right)^{1/2}}_{(*)}.$$

By Remark 4, the expression $(*)$ is bounded by some constant only depending on the norm of f_γ . Hence, $1 \lesssim \phi'(x)|\gamma - \gamma'|$. But, by Lemma 3, this implies that $d_\phi(\gamma, \gamma') \gtrsim 1$. \square

Lemma 11. *Let E be a Hermite-Biehler function with phase function ϕ . If $\phi'(x)dx$ is a locally doubling measure on \mathbb{R} , then for every sampling sequence Λ , there exists a ϕ -separated sequence $\Lambda' \subset \Lambda$ which is also a sampling sequence.*

This result can be deduced from Lemma 9. For a detailed proof in the Bergman case see [14, p. 201].

3. BASIC CONSTRUCTIONS: MULTIPLIER AND PEAK FUNCTIONS

In this section, we consider a larger class of spaces than the de Branges spaces. In view of this, let Ψ denote a real valued subharmonic function with Laplacian supported on the real line, and define the space

$$\mathcal{F}_\Psi^\infty = \{f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}_\Psi^\infty} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\Psi(z)} < \infty\}.$$

Note that if E is a Hermite-Biehler function and, as above, we define

$$\Phi(z) := \begin{cases} \log |E(z)| & \text{if } \operatorname{Im} z > 0, \\ \log |E^*(z)| & \text{if } \operatorname{Im} z < 0. \end{cases}$$

then $f \in \mathcal{F}_\Phi^\infty$ if and only if f/E and f^*/E are in $H^\infty(\mathbb{C}_+)$, with the norm computed on \mathbb{R} .

Let μ denote the measure on \mathbb{R} given by $\Delta\Psi$. We make the additional assumptions that this measure is doubling, and that $\mu = \psi'(t)dt$ for some suitable function $\psi \in C^\infty(\mathbb{R})$ satisfying (b) of Lemma 2, i.e., there exist constants such that the following local doubling property holds:

$$\psi'(x) \simeq \psi'(y) \text{ whenever } |\psi(x) - \psi(y)| \leq 1. \quad (8)$$

Recall that by d_ψ , I_ψ and D_ψ we denote the metric, open intervals and open balls induced by the measure $\psi'(t)dt$, respectively, and that by $Z(f)$ we denote the zero set of the entire function f .

3.1. Multiplier construction. We first establish the existence of the so-called multiplier function. We follow the proof of Theorem 41 in [19] closely.

Theorem 3. *Let Ψ, ψ and μ be as above. If the sequence $\Lambda = (\lambda_j)$ of real numbers is separated in the metric d_ψ and satisfies*

$$D_\psi^+(\Lambda) := \limsup_{r \rightarrow \infty} \sup_{\mu(I)=r} \frac{|I \cap \Lambda|}{r} < \frac{1}{2\pi}, \quad (9)$$

then we can construct a function $f \in \mathcal{F}_\Psi^\infty$ such that $\Lambda \subset Z(f)$, and, for some number $\eta > 0$, it holds that:

- (a) *The set Λ is separated from the other zeroes of f in the sense that $D_\psi(\lambda, \eta) \cap Z(f) = \{\lambda\}$ for all $\lambda \in \Lambda$.*
- (b) *There exist constants such that $|f(x)e^{-\Psi(x)}| \simeq d_\psi(x, \lambda)$ for all $\lambda \in \Lambda$ and $x \in I_\psi(\lambda, \eta)$.*

The idea in the first part of the proof, is to use (9) to associate with $\mu = \psi'(t)dt$ a measure on \mathbb{C} that can be split up into pieces with good cancelation properties. To this end, we need the following lemma.

Lemma 12. *Let τ be a complex measure supported in an interval $I \subset \mathbb{R}$ with $|\tau(I)| \geq 1$ and $\int_I |x|^j d|\tau|(x) < \infty$ for $j = 1, \dots, n$. Then there exists a set of points $\{\xi_1, \dots, \xi_n\} \in \mathbb{C}$ such that*

$$\frac{1}{\tau(I)} \int x^j d\tau(x) = \frac{1}{n} \sum_{k=1}^n \xi_k^j,$$

for $j \in \{1, \dots, n\}$. Moreover, if $I = I(c, r)$, the points $\{\xi_1, \dots, \xi_n\}$ are in a ball $D(c, Cr)$ for some constant C depending only on n and $|\tau|(I)$.

Proof. We sketch a proof due to Ortega-Cerdà [23]. Let

$$p_j(z_1, \dots, z_n) = z_1^j + \dots + z_n^j, \quad j = 1, \dots, n.$$

Our objective is to solve the system of equations

$$p_j(z_1, \dots, z_n) = \frac{n}{\tau(I)} \int x^j d\tau(x) = c_j, \quad j = 1, \dots, n.$$

We first suppose that $I = (-1, 1)$. Observe that $|c_j| \leq n|\tau|(I)$. The polynomials p_1, \dots, p_n are called the power sum symmetric polynomials, and it is well-known that they generate the ring of symmetric polynomials with rational coefficients in n variables. In particular, the coefficients of the polynomial

$$\prod_{i=1}^n (t - z_i) = t^n - e_1(z_1, \dots, z_n)t^{n-1} + \dots + (-1)^n e_n(z_1, \dots, z_n),$$

which are the elementary symmetric polynomials, are in the ring of symmetric polynomials. In view of this, there exist polynomials $Q_j(z_1, \dots, z_n)$ such that

$$e_j = Q_j(p_1, \dots, p_n), \quad j = 1, \dots, n.$$

As the elementary symmetric polynomials e_j also generate the ring of symmetric polynomials in n variables, there exist polynomials $R_j(z_1, \dots, z_n)$ such that

$$p_j = R_j(e_1, \dots, e_n), \quad j = 1, \dots, n.$$

Since the power sum symmetric polynomials are algebraically independent over \mathbb{Q} [18], it now follows that

$$R_j(Q_1(z_1, \dots, z_n), \dots, Q_n(z_1, \dots, z_n)) = z_j, \quad j = 1, \dots, n.$$

Let $\{\xi_1, \dots, \xi_n\}$ be the roots of the polynomial

$$t^n - Q_1(c_1, \dots, c_n)t^{n-1} + \dots + (-1)^n Q_n(c_1, \dots, c_n).$$

Clearly,

$$e_j(\xi_1, \dots, \xi_n) = Q_j(c_1, \dots, c_n), \quad j = 1, \dots, n,$$

and therefore

$$\begin{aligned} p_j(\xi_1, \dots, \xi_n) &= R_j(e_1(\xi_1, \dots, \xi_n), \dots, e_n(\xi_1, \dots, \xi_n)) \\ &= R_j(Q_1(c_1, \dots, c_n), \dots, Q_n(c_1, \dots, c_n)) = c_j. \end{aligned}$$

As it holds that

$$|\xi_j| \leq \max_{j=1, \dots, n} \{|Q_j(c_1, \dots, c_n)|\} + 1, \quad j = 1, \dots, n,$$

the points ξ_j are in $D(0, C)$ for some constant $C > 0$ depending only on n and $|\tau|(I)$.

In the general case, the interval $I(c, r) = (c - r, c + r)$ can be sent to $(-1, 1)$, by an affine map. If we apply the above procedure to the measure $\tilde{\tau}$ on $(-1, 1)$ induced by τ , we obtain a set of points $\tilde{\xi}_k$. With this, it is not hard to see that the points

$$\xi_k = r\tilde{\xi}_k + c, \quad k = 1, \dots, n,$$

have the desired properties. Indeed,

$$\begin{aligned} \frac{1}{\tau(I)} \int_I x^j d\tau(x) &= \frac{1}{\tau(I)} \int_{-1}^1 (rx + c)^j d\tilde{\tau}(x) = \sum_{\ell=0}^j \binom{j}{\ell} \frac{c^{j-\ell} r^\ell}{\tau(I)} \int_{-1}^1 x^\ell d\tilde{\tau}(x) \\ &= \sum_{\ell=0}^j \binom{j}{\ell} c^{j-\ell} r^\ell \frac{1}{n} \sum_{k=1}^n \tilde{\xi}_k^\ell = \frac{1}{n} \sum_{k=1}^n \xi_k^j. \end{aligned}$$

□

Proof of Theorem 3. Given $x \in \mathbb{R}$, we set $\rho_x = 1/\psi'(x)$. Observe that in this notation, $\mu(D(x, \rho_x)) \simeq 1$. Define the measure

$$\tilde{\mu} := \frac{\mu}{2\pi} - \sum_{\lambda \in \Lambda} \delta_\lambda.$$

Fix $n \in \mathbb{N}$ so large that $n\gamma > 1$, where γ is the exponent appearing in Lemma 1 for μ .

By condition (9), given any $\epsilon > 0$, if we choose $M > 0$ large enough, then it holds for any interval $I \subset \mathbb{R}$, with $\mu(I) = 2\pi M$, that

$$\epsilon M \leq \tilde{\mu}(I) \leq M.$$

Choose M so large that both $M\epsilon \geq n^2$ and the above holds, and let $\{I_k\}$ be a partition of \mathbb{R} into intervals with $\mu(I_k) = 2\pi M$. Note that if M is an integer, then so is $\tilde{\mu}(I_k)$. Hence, without loss of generality, we can add points to the set Λ , while keeping the separation (since no more than M points have to be added to any single interval I_k), so that

$$\tilde{\mu}(I_k) = n^2, \quad \forall k \in \mathbb{Z}.$$

Let S_k be the square, symmetric with respect to \mathbb{R} , such that $S_k \cap \mathbb{R} = I_k$, and consider the measure $\tilde{\mu}$ as a singularly supported measure on \mathbb{C} . Let $\tilde{\mu}_k$ denote its restriction to S_k .

We apply Lemma 12 to each of the measures $\tilde{\mu}_k$ to find point sets $\Xi_k = \{\xi_{k,1}, \dots, \xi_{k,n}\} \subset CS_k$ for which the measures

$$\tilde{\mu}_k - n \sum_{\xi \in \Xi_k} \delta_\xi$$

each have n vanishing moments. Here CS_k is the square co-centric with S_k and with side-lengths scaled up by the factor C .

The next step of the proof is to modify each Ξ_k to obtain sets Σ_k such that for each $k \in \mathbb{Z}$, the measure

$$\nu_k = \tilde{\mu}_k - \sum_{\sigma \in \Sigma_k} \delta_\sigma$$

also has n vanishing moments, while keeping the set Λ separated from $\Sigma := \cup \Sigma_k$ in the sense that there exists an $\eta > 0$ such that for all $\lambda \in \Lambda$, the discs $D_\psi(\lambda, \eta)$ have empty intersection with Σ . In this notation, the set Σ_k may have some points of multiplicity n .

Because of the separation of Λ , shrinking $\eta > 0$ if necessary, we can suppose that the euclidean discs $D(\lambda, \eta\rho_\lambda)$ are pairwise disjoint for $\lambda \in \Lambda$.

Now, for each $\xi \in \Xi_k$ belonging to $D(\lambda, \eta\rho_\lambda/5)$ for some $\lambda \in \Lambda$, set

$$\Sigma_\xi = \{\xi + (3\eta\rho_\lambda/5)e^{\frac{2\pi il}{n}} : 0 \leq l < n\}.$$

Let $\tilde{\Sigma}_k$ be the union of all such sets for those $\xi \in \Xi_k$. With this, define

$$\Sigma_k = \tilde{\Sigma}_k \cup \{\xi \in \Xi_k : \xi \notin D(\lambda, \eta\rho_\lambda), \forall \lambda \in \Lambda\}.$$

As mentioned above, in this definition we allow some points to have multiplicity n . By construction, the set Σ is clearly separated from Λ in the sense given above (see Figure 1). To see that the measure ν_k also has n vanishing moments, it suffices to observe that for any $z, \tau \in \mathbb{C}$ and $0 \leq j < n$ we have

$$\sum_{l=0}^{n-1} (z + \tau e^{\frac{2\pi il}{n}})^j = nz^j.$$

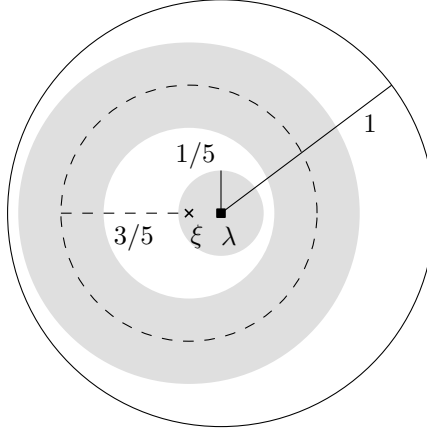


FIGURE 1. We illustrate how Σ_k is obtained by replacing ξ by points placed on a circle with center ξ and radius depending on the proximity to the closest λ . The dotted circle indicates the placement of the n new points, while the shaded area indicates the new separation between the new points and λ . (Note that we have normalized the distances.)

Note that it may happen that the points in Σ_k are now contained in some square larger than CS_k . However, since ψ satisfies the local doubling property (8), it is easily seen that this may be remedied by increasing C slightly, independent of k .

To define the desired multiplier function f , let $\nu = \sum \nu_k$, and set

$$w(z) := \int_{\mathbb{C}} \log |z - \zeta| d\nu(\zeta).$$

In particular, $\Delta w = 2\pi\nu$. Assume for a moment that $w(z)$ is finite for all $z \in \mathbb{C}$, and let g denote any entire function with zeroes $\Sigma \cup \Lambda$ with appropriate multiplicities. Then $\Delta w = \Delta(\Psi - \log |g|)$. I.e.,

$$\Delta(w - \Psi + \log |g|) = 0.$$

Hence, by Weyl's lemma, the function $w - \Psi + \log |g|$ is harmonic on \mathbb{C} . Therefore, there exists an entire function h of which this is the real part, so $w = \operatorname{Re} h - \log |g| + \Psi$. We set

$$f := e^{-h} g.$$

The final step of the proof is to obtain the required estimates for f . Indeed, to show that $f \in \mathcal{F}_{\Psi}^{\infty}$ and that (b) holds, it suffices to find positive constants $B, C, D < \infty$ for which

$$-w(z) \leq B, \quad \forall z \in \mathbb{C}, \quad (10)$$

and for small enough $\eta > 0$ and all $\lambda \in \Lambda$,

$$-C \leq w(z) - \log \frac{R\rho_{\lambda}}{|z - \lambda|} \leq D, \quad \forall z \in D(\lambda, \eta\rho_{\lambda}). \quad (11)$$

For the estimates below, we let x_k denote the center of I_k and set $\rho_{x_k} = \rho_k$. By the local doubling property (8), it follows readily that there is a radius $R > 0$ such that for all $k \in \mathbb{Z}$ we have $CS_{k-1} \cup CS_k \cup CS_{k+1} \subset D(x_k, R\rho_k)$.

We first prove that (10) holds. Fix $z \notin \cup_k D(x_k, 2R\rho_k)$ and let $\zeta \in D(x_k, R\rho_k)$. In this case, $|\zeta - x_k| \leq R\rho_k \leq |z - x_k|$. By $P_k(\zeta) = P_{k,n-1,z}(\zeta)$, denote the Taylor polynomial of order $n-1$ of the function $\log(z - \zeta)$ at the point x_k . Then, since $|z - \zeta| \geq |z - x_k|/2$, it holds that

$$|\log(z - \zeta) - P_k(\zeta)| \leq \max_{\xi \in \langle x_k, \zeta \rangle} \left| \frac{(\zeta - x_k)^n}{(z - \xi)^n n} \right| \lesssim \frac{\rho_k^n}{|z - x_k|^n},$$

where $\langle x_k, \zeta \rangle$ is the segment in \mathbb{C} with endpoints x_k and ζ . So, by the n vanishing moments of ν_k ,

$$\begin{aligned} \left| \int_{\mathbb{C}} \log |z - \zeta| d\nu(\zeta) \right| &= \left| \operatorname{Re} \int_{\mathbb{C}} \log(z - \zeta) d\nu(\zeta) \right| \leq \left| \int_{\mathbb{C}} \log(z - \zeta) d\nu(\zeta) \right| \\ &\leq \left| \sum_{k \in \mathbb{Z}} \int_{CS_k} \log(z - \zeta) d\nu_k(\zeta) \right| \leq \sum_{k \in \mathbb{Z}} \left| \int_{CS_k} (\log(z - \zeta) - P_k(\zeta)) d\nu_k(\zeta) \right| \\ &\lesssim \sum_{k \in \mathbb{Z}} \frac{\rho_k^n}{|z - x_k|^n} \int_{CS_k} d|\nu_k|(\zeta) \lesssim \sum_{k \in \mathbb{Z}} \frac{\rho_k^n}{|z - x_k|^n}. \end{aligned}$$

Let $k_z \in \mathbb{N}$ be such that $\operatorname{Re} z \in I_{k_z}$. To see that the sum converges with a bound independent of z , it suffices to establish that $|z - x_k| \geq A\rho_k(|k - k_z|^\gamma + 1)$, for some constant A only depending on R . Recall that $n\gamma > 1$, and γ is the exponent appearing in Lemma 1 for μ . We leave it to the reader to find that such an estimate follows from the doubling property of μ .

Next, suppose that $z \in D(x_{k_z}, 2R\rho_{k_z})$ for some $k_z \in \mathbb{N}$. By the doubling property, there exists a number q , such that for $|k - k_z| \geq q$, then $z \notin D(x_k, 2R\rho_k)$. So, if we make the split

$$w(z) = \underbrace{\sum_{|k-k_z| \leq q} \int_{\mathbb{C}} \log |z - \zeta| d\nu_k(\zeta)}_{:=w_1(z)} + \underbrace{\sum_{|k-k_z| > q} \int_{\mathbb{C}} \log |z - \zeta| d\nu_k(\zeta)}_{:=w_2(z)},$$

then we can use the same method to get a bound $|w_2(z)| \leq B$ as we did in the first case above. To estimate $w_1(z)$, we increase R if necessary so that $\cup_{|k-k_z| \leq q} CS_k \subset D(x_{k_z}, R\rho_{k_z})$ (this is possible independently of k). Then we can use the first vanishing moment of ν_k to get

$$\begin{aligned} -w_1(z) &= \sum_{|k-k_z| \leq q} \int_{CS_k} \log \frac{R\rho_{k_z}}{|z - \zeta|} d \left(\frac{\mu_k}{2\pi} - \sum_{\lambda \in \Lambda \cap I_k} \delta_\lambda - \sum_{\sigma \in \Sigma_k} \delta_\sigma \right) (\zeta) \\ &\lesssim \int_{D(x_{k_z}, R\rho_{k_z})} \log \frac{R\rho_{k_z}}{|z - \zeta|} d\mu(\zeta). \end{aligned}$$

Since $d\mu = \psi'(t)dt$, and $\psi'(t) \simeq \psi'(x_{k_z})$ for $t \in I(x_{k_z}, R\rho_{k_z})$, it follows by basic calculus that the integral on the right-hand side is comparable to R .

We turn to the demonstration of (11). Consider $\lambda \in \Lambda$ and $z \in D(\lambda, \eta\rho_\lambda)$, where $\eta > 0$ is a separation constant of Λ from Σ . To get the lower estimate, we make the same split $w(z) = w_1(z) + w_2(z)$ as above, using λ in place of k_z . The estimate of $w_2(z)$ follows as before with the obvious modifications. In the lower estimate of $w_1(z)$, recall that above the expression

$$I := \sum_{|k-k_z| \leq q} \int_{CS_k} \log \frac{R\rho_\lambda}{|z-\zeta|} d \left(\sum_{\lambda \in \Lambda \cap S_k} \delta_\lambda + \sum_{\sigma \in \Sigma_k} \delta_\sigma \right) (\zeta)$$

was estimated simply as $I \geq 0$. The lower estimate of $w_1(z)$ follows in the current case, if we replace this by

$$I = \sum_{|k-k_z| \leq q} \sum_{\gamma \in (\Lambda \cap I_k) \cup \Sigma_k} \log \frac{R\rho_\lambda}{|z-\gamma|} \geq \log \frac{R\rho_\lambda}{|z-\lambda|}.$$

As for the upper estimate of $w_1(z)$, since μ is a positive measure,

$$w_1(z) \leq \log \frac{R\rho_\lambda}{|z-\lambda|} + \sum_{|k-k_z| \leq q} \sum_{\gamma \in \Sigma_k \cup \Lambda \cap I_k \setminus \{\lambda\}} \log \frac{R\rho_\lambda}{|z-\gamma|}.$$

It follows easily that the whole sum is bounded by some uniform constant, since every γ on the right-hand side is at least at a euclidean distance of $\eta\rho_\lambda/2$ from z . This ends the proof. \square

3.2. Peak functions. As an application of the above multiplier construction, we obtain a class of peak functions analogue to the ones used by Beurling in his original work on sampling sequences. First, we record the following simple lemma.

Lemma 13. *Let Ψ and ψ be as above. Then, for every $x_0 \in \mathbb{R}$ there exist a function $f \in \mathcal{F}_\Psi^\infty$ and a sequence $\Lambda \subset Z(f) \cap \mathbb{R}$ that satisfy the following conditions, all with constants not depending on x_0 :*

- (i) *The set $\Lambda \cup \{x_0\}$ is separated.*
- (ii) *The quantity $|f(x_0)|e^{-\Psi(x_0)}$ is bounded below.*
- (iii) *The map $x \mapsto d_\psi(x, \Lambda)$ is bounded above.*
- (iv) *We have $|f(x)| \lesssim d_\psi(x, \Lambda)e^{\Psi(x)}$ for every $x \in \mathbb{R}$.*

Proof. Set $\Lambda = \{\lambda \in \mathbb{R} : d_\psi(x_0, \lambda) = 4\pi k, 0 \neq k \in \mathbb{Z}\}$. Since $D_\psi^+(\Lambda) = 1/4\pi$ we may use these Λ to repeat the construction from Theorem 3. We do this while keeping the sequence Σ separated not only from Λ but also from $\Lambda \cup \{x_0\}$. This can be done in the same way as in the proof of Theorem 3, with a constant of separation not depending on x_0 . Note that the constants B, C and D appearing in (10) and (11) depend only on the distribution of Λ (with respect to d_ψ) and the doubling constant of $\mu = \psi'(x)dx$. Hence, the bounds obtained in these constructions do not depend on x_0 . Properties (i), (iii) and (iv) follow immediately. To see that property (ii) is satisfied we make an estimate similar to the one on the right side of

(11), with $z = x_0$, and use the fact that x_0 is uniformly separated from all points in $\Lambda \cup \Sigma$. \square

We are now ready to construct our peak functions.

Lemma 14. *Let Ψ and ψ be as above, and fix $a > 0$ and $k \in \mathbb{N}$. For every $x \in \mathbb{R}$ there is a function $h(x, \cdot) \in \mathcal{F}_{a\Psi}^\infty$ that satisfies $h(x, x) = 1$ and*

$$|h(x, y)| \leq C \frac{1}{1 + d_\psi^k(x, y)} e^{a(\Psi(y) - \Psi(x))} \quad \forall y \in \mathbb{R}, \quad (12)$$

where C is a constant not depending on x . Moreover, the following estimate holds whenever k is big enough:

$$\sup_{y \in \mathbb{R}} \psi'(y) \int_{\mathbb{R}} |h(x, y)|^2 e^{2a(\Psi(x) - \Psi(y))} dx < \infty. \quad (13)$$

Proof. For $x_0 \in \mathbb{R}$, let $f \in \mathcal{F}_{a\Psi}^\infty$ be the function from Lemma 13. Property (iii) of the lemma implies that there exists $L > 0$, not depending on x_0 , such that the interval $I_\psi(x_0, L)$ includes at least M points from the corresponding sequence Λ . Choose such $\sigma_1, \dots, \sigma_M \in \Lambda$ and set

$$h(x_0, z) = \frac{f(z)}{(z - \sigma_1) \cdots (z - \sigma_M)} \frac{c(x_0) e^{-\varepsilon \Psi(x_0)}}{(\phi'(x_0))^M}$$

where $c(x_0)$ is chosen to satisfy $h(x_0, x_0) = 1$. It is clear that $h(x_0, z) \in \mathcal{F}_{a\Psi}^\infty$ since outside of a big enough disc we have $|h(x_0, z)| \lesssim |f(z)|$. The inequality (12) is satisfied whenever M is big enough, as in the proof of [19, Theorem 18]. To prove (13), we use (12) to see that for $x, y \in \mathbb{R}$ we have

$$\begin{aligned} \psi'(y) \int_{\mathbb{R}} |h(x, y)|^2 e^{2a(\Psi(x) - \Psi(y))} dx &\lesssim \int_{\mathbb{R}} \frac{\psi'(y) dx}{1 + d_\psi^{2k}(x, y)} \\ &= \left[\int_{I_\psi(y, 1)} + \int_{\mathbb{R}/I_\psi(y, 1)} \right] \frac{\psi'(y) dx}{1 + d_\psi^{2k}(x, y)}, \end{aligned}$$

where the constant does not depend on y . For the first integral on the right-hand side, the needed estimate is easily obtained. Next, since $\psi'(x)dx$ is a doubling measure, by Remark 2, there exists a $\delta > 0$ such that the second integral is dominated by

$$\int_{\mathbb{R}} \frac{d_\psi^\delta(x, y) \psi'(x) dx}{1 + d_\psi^{2k}(x, y)} \lesssim \int_{\mathbb{R}} \frac{\psi'(x) dx}{1 + d_\psi^{2k-\delta}(x, y)}.$$

By the change of variables $t = \psi(x) - \psi(y)$, this is seen to converge for large k . \square

4. PROOFS OF MAIN RESULTS

4.1. Necessary conditions for sampling and interpolation. To get the necessary conditions for a real sequence of points to be interpolating or sampling, we use the technique developed by Ramanathan and Steger [25] (see also [24, Lemma

4]). Since we have canonical orthonormal bases in de Branges spaces, this theorem is easily applicable.

Theorem 4. *Let E be a Hermite-Biehler function with phase function ϕ , and suppose that $\phi'(x)dx$ is a doubling measure on \mathbb{R} .*

If $\Gamma = \{\gamma_k\} \subset \mathbb{R}$ is a separated sampling sequence for $H(E)$ and $\Lambda = \{\lambda_k\} \subset \mathbb{R}$ is an interpolating sequence for $H(E)$, then $D_\phi^+(\Lambda) \leq D_\phi^+(\Gamma)$ and $D_\phi^-(\Lambda) \leq D_\phi^-(\Gamma)$.

Proof. Recall that K_z is the reproducing kernel of $H(E)$ at the point $z \in \mathbb{C}$, while $\tilde{K}_z = K_z / \|K_z\|_{H(E)}$. We denote by $\{G_k\}$ and $\{L_k\}$ the dual frame of $\{\tilde{K}_{\gamma_k}\}$, and the biorthogonal basis of $\{\tilde{K}_{\lambda_k}\}$, respectively.

For $t, r > 0$, $x \in \mathbb{R}$ we define the sets

$$W_\Gamma(x) = \text{span}\{G_k : \gamma_k \in I_\phi(x, t+r)\}, \quad W_\Lambda(x) = \text{span}\{\tilde{K}_{\lambda_k} : \lambda_k \in I_\phi(x, t)\},$$

and the corresponding orthogonal projections P_Γ and P_Λ onto $W_\Gamma(x)$ and $W_\Lambda(x)$, respectively.

The idea of the proof is the following observation. It is clearly sufficient to show that, given any $\epsilon > 0$, then for $r > 0$ big enough, and all $t, x \in \mathbb{R}$, the trace of the operator $T = P_\Lambda P_\Gamma P_\Lambda$, as an operator on $W_\Lambda(x)$, satisfies

$$(1 - \epsilon)|\Lambda \cap I_\phi(x, t)| \leq \text{tr}(T) \leq |\Gamma \cap I_\phi(x, t+r)|. \quad (14)$$

On the finite dimensional space $W_\Lambda(x)$, the trace of the self-adjoint and positive operator T is computed by

$$\text{tr}(T) = \sum_{\lambda_k \in I_\phi(x, t)} \langle T \tilde{K}_{\lambda_k}, L_k \rangle.$$

The upper estimate follows easily from the fact that eigenvalues of T are bounded by one in modulus:

$$\text{tr}(T) \leq \text{rank}(T) \leq \dim W_\Gamma(x) \leq |\Gamma \cap I_\phi(x, t+r)|.$$

For the lower estimate, we use the Cauchy-Schwartz inequality and the biorthogonality property of $\{L_k\}$ to get

$$\langle T \tilde{K}_{\lambda_k}, L_k \rangle \geq 1 - \|P_\Gamma \tilde{K}_{\lambda_k} - \tilde{K}_{\lambda_k}\|_{H(E)} \|P_\Lambda L_k\|_{H(E)}.$$

The term $\|P_\Lambda L_k\|_{H(E)}$ is uniformly bounded, therefore it only remains to make $\|P_\Gamma \tilde{K}_{\lambda_k} - \tilde{K}_{\lambda_k}\|_{H(E)}$ small. Since Γ is sampling, a simple computation involving the projection P_Γ yields

$$\|P_\Gamma \tilde{K}_{\lambda_k} - \tilde{K}_{\lambda_k}\|_{H(E)}^2 \lesssim \sum_{\gamma_n \notin I_\phi(x, t+r)} |\langle \tilde{K}_{\lambda_k}, \tilde{K}_{\gamma_n} \rangle|^2.$$

Now let $\lambda_k \in I_\phi(x, t)$, $\gamma_n \notin I_\phi(x, t+r)$ and let $\delta > 0$ be the separation constant of Γ with respect to the metric d_ϕ . Inserting the expression of our reproducing

kernels, and making a simple estimate, this is bounded above by

$$\frac{1}{\phi'(\lambda_k)} \sum_{\gamma_n \notin I_\phi(x, t+r)} \int_{I_\phi(\gamma_n, \delta)} \frac{ds}{(\lambda_k - s)^2} \lesssim \frac{1}{\phi'(\lambda_k)} \int_{\mathbb{R} \setminus I_\phi(\lambda_k, r/2)} \frac{ds}{(\lambda_k - s)^2} \lesssim \frac{1}{r^{1/\alpha}}$$

The left-most inequality holds as soon as $r > 2\delta$, while the right-most holds for the $\alpha > 1$ given by Lemma 3. With this, (14) follows. \square

4.2. Sufficiency for interpolation. For a de Branges space $H(E)$ with Φ defined as in (2), we recall that $\Delta\Phi = 2\phi'(x)dx$. So with $\Psi = \Phi$, given a ϕ -separated sequence $\Lambda \subset \mathbb{R}$ for which $D_\phi^+(\Lambda) < 1/\pi$, it follows from Theorem 3 that for small enough $\eta > 0$ there exists a function $f \in \mathcal{F}_\Phi^\infty$ such that uniformly in $\lambda \in \Lambda$ we have

$$|f'(\lambda)| \simeq |E(\lambda)|\phi'(\lambda). \quad (15)$$

and

$$|(f/E)(x)/(x - \lambda)| \simeq \phi'(\lambda), \quad x \in I_\phi(\lambda, \eta) \quad (16)$$

We now use this multiplier f to solve the interpolation problem $F(\lambda) = w_\lambda$, where the data (w_λ) satisfies $\sum_{\lambda \in \Lambda} |w_\lambda|^2 / |E(\lambda)|^2 \phi'(\lambda) < \infty$. Our objective is to show that the solution is given by the following Lagrange-type interpolation function:

$$F(z) = \sum_{\lambda \in \Lambda} w_\lambda \frac{f(z)}{(z - \lambda)f'(\lambda)}$$

We proceed by duality. First,

$$\|F\|_{H(E)} = \sup_{\substack{h \in H^2(\mathbb{C}_+) \\ \|h\|_2=1}} \left| \sum \frac{w_\lambda}{f'(\lambda)} \int_{\mathbb{R}} \overline{h(t)} \frac{f(t)/E(t)}{t - \lambda} dt \right|.$$

At each term of this sum, we multiply and divide by $\sqrt{\phi'(\lambda)}|E(\lambda)|$.

The Cauchy-Schwarz inequality, in combination with the assumption on $\{w_\lambda\}$ and (15), yields

$$\|F\|_{H(E)}^2 \lesssim \sup_{\substack{h \in H^2(\mathbb{C}_+) \\ \|h\|_2=1}} \sum_{\lambda} \frac{1}{\phi'(\lambda)} \left| H \left(\frac{\bar{h}f}{E} \right) (\lambda) \right|^2, \quad (17)$$

where H denotes the Hilbert transform on the real line.

Given $\lambda \in \Lambda$, we consider the decomposition

$$H \left(\frac{\bar{h}f}{E} \right) (\lambda) = \left(\int_{d_\phi(t, \lambda) < \eta} + \int_{d_\phi(t, \lambda) > \eta} \right) \overline{h(t)} \frac{f(t)/E(t)}{t - \lambda} dt := A + B. \quad (18)$$

With this, the Cauchy-Schwarz inequality, Lemma 3 and (16), imply

$$|A|^2 \lesssim \phi'(\lambda) \int_{d_\phi(t, \lambda) < \eta} |h(t)|^2 dt. \quad (19)$$

As for B , we disconnect the λ appearing in the domain of integration and in the function, and define

$$B(\xi) := \int_{d_\phi(t, \lambda) > \eta} \overline{h(t)} \frac{f(t)/E(t)}{t - \xi} dt.$$

This function is analytic for $\xi \in D_\phi(\lambda, \eta)$. In a similar way we define the function $A(\xi)$, which is, however, only analytic for $\xi \notin \mathbb{R}$. By subharmonicity, and since the radius of $D_\phi(\lambda, \eta/2)$ is comparable to $1/\phi'(\lambda)$, we get the inequality

$$\begin{aligned} |B|^2 &\lesssim \phi'(\lambda)^2 \int_{D_\phi(\lambda, \eta/2)} |B(\xi)|^2 dm(\xi) \\ &\leq \phi'(\lambda)^2 \int_{D_\phi(\lambda, \eta/2)} \left| H \left(\frac{\bar{h}f}{E} \right) (\xi) \right|^2 dm(\xi) + \underbrace{\phi'(\lambda)^2 \int_{D_\phi(\lambda, \eta/2)} |A(\xi)|^2 dm(\xi)}_{:=C}. \end{aligned}$$

Since f/E is bounded on \mathbb{R} , and the Hilbert transform is bounded on $L^2(\mathbb{R})$, we use the fact that $\phi'(\lambda)\chi_{D_\phi(\lambda, \eta/2)}$ gives a Carleson measure for both $H^2(\mathbb{C}_+)$ and $H^2(\mathbb{C}_-)$, with constant only depending on η , to see that

$$\phi'(\lambda)C \lesssim \left\| \frac{\bar{h}f}{E} \chi_{I_\phi(\lambda, \eta)} \right\|_{L^2(\mathbb{R})}^2 \lesssim \int_{I_\phi(\lambda, \eta)} |h(t)|^2 dt.$$

Plugging these estimates into (18), yields

$$\left| H \left(\frac{\bar{h}f}{E} \right) (\lambda) \right|^2 \lesssim \phi'(\lambda)^2 \int_{D_\phi(\lambda, \eta/2)} \left| H \left(\frac{\bar{h}f}{E} \right) (\xi) \right|^2 dm(\xi) + \phi'(\lambda) \int_{d_\phi(t, \lambda) < \eta} |h(t)|^2 dt.$$

Due to the ϕ -separation of the points λ we get

$$\sum_{\lambda \in \Lambda} \int_{d_\phi(t, \lambda) < \eta} |h(t)|^2 dt \leq \|h\|_{L^2(\mathbb{R})}^2.$$

In addition, by Lemma 6,

$$\sum_{\lambda \in \Lambda} \phi'(\lambda) \int_{D_\phi(\lambda, \eta/2)} \left| H \left(\frac{\bar{h}f}{E} \right) (\xi) \right|^2 dm(\xi) \lesssim \left\| H \left(\frac{\bar{h}f}{E} \right) \right\|_{H^2(\mathbb{C}_+)}^2 + \left\| H \left(\frac{\bar{h}f}{E} \right) \right\|_{H^2(\mathbb{C}_-)}^2.$$

This yields the desired conclusion as the Cauchy transform is bounded on $L^2(\mathbb{R})$, and f/E is bounded on \mathbb{R} .

4.3. Sufficiency for sampling. We follow the ideas of Beurling in the case of the Paley-Wiener space [7], as adapted to the Fock space setting by Marco, Massaneda and Ortega-Cerdà in [19].

For the space \mathcal{F}_Ψ^∞ , defined in Section 3, we say that a sequence $\Lambda \subseteq \mathbb{R}$ is a sampling sequence if there exists some constant $C > 0$ such that for all $f \in \mathcal{F}_\Psi^\infty$ we have

$$\|f\|_{\mathcal{F}_\Psi^\infty} \leq C \sup_{\lambda \in \Lambda} |f(\lambda)| e^{-\Psi(\lambda)}.$$

The first lemma states that in order to prove that a sequence is sampling in a de Branges space, it is enough to prove that it is sampling in such a uniform norm space.

Lemma 15. *Let E be a Hermite-Biehler function with phase function ϕ , suppose that $\phi'(x)dx$ is a doubling measure on \mathbb{R} and let Φ be defined by (2). Moreover, suppose that Λ is a ϕ -separated sequence.*

If, for some $\epsilon > 0$, the sequence Λ is a sampling sequence for $\mathcal{F}_{(1+\epsilon)\Phi}^\infty$, then it is a sampling sequence for $H(E)$.

Proof. It is sufficient to show that if Λ is a sampling sequence for $\mathcal{F}_{(1+\epsilon)\Phi}^\infty$, then every $g \in H(E)$ is pointwise equal to a series of the form $\sum a_\lambda \tilde{K}_\lambda$ with $\sum |a_\lambda|^2 \lesssim \|g\|_{H(E)}^2$, where K_w is the reproducing kernel in $H(E)$ at $w \in \mathbb{R}$.

So, assume that for some $\epsilon > 0$, the sequence Λ is a sampling sequence for $\mathcal{F}_{(1+\epsilon)\Phi}^\infty$. Following a duality argument in [7, pp. 348-358], there exist functions $g_\lambda(x)$, with $\sum_{\lambda \in \Lambda} |g_\lambda(x)| \leq K$ uniformly in x , so that for all $f \in \mathcal{F}_{(1+\epsilon)\Phi}^\infty$ such that $|f(x)|e^{-(1+\epsilon)\Phi(x)} \rightarrow 0$ as $|x| \rightarrow \infty$, we have

$$f(x)e^{-(1+\epsilon)\Phi(x)} = \sum_{\lambda \in \Lambda} f(\lambda)e^{-(1+\epsilon)\Phi(\lambda)} g_\lambda(x) \quad \forall x \in \mathbb{R}. \quad (20)$$

Let $h(x, \cdot)$ be the function from Lemma 14 (corresponding to the ϵ from above), and set $f_{w,x}(z) = K_w(z)h(x, z)$. Clearly, $|f_{w,x}(y)|e^{-(1+\epsilon)\Phi(y)} \rightarrow 0$ as $|y| \rightarrow \infty$, and so (20) holds for this function. In particular, we have $h(x, x) = 1$, so the formula gives pointwise

$$K_w(x)e^{-\Phi(x)} = \sum_{\lambda \in \Lambda} e^{\epsilon(\Phi(x)-\Phi(\lambda))} \sqrt{\phi'(\lambda)} h(x, \lambda) g_\lambda(x) \tilde{K}_\lambda(w), \quad \forall x \in \mathbb{R}. \quad (21)$$

Moreover, by the estimate (13) of Lemma 14, the functions

$$\zeta_\lambda(x) = e^{\epsilon(\Phi(x)-\Phi(\lambda))} \sqrt{\phi'(\lambda)} h(x, \lambda) g_\lambda(x)$$

belong to $L^2(\mathbb{R})$.

Now, to show that the series in (21) converges in $L^2(\mathbb{R})$, it suffices to verify that the sequence $\{\zeta_\lambda\}$ is a Bessel sequence in $L^2(\mathbb{R})$ and that the sequence $\{\tilde{K}_\lambda(w)\}_{\lambda \in \Lambda}$, for fixed w , is in $l^2(\Lambda)$. The last claim follows from Corollary 2, while the first holds as $\sum_{\lambda \in \Lambda} |g_\lambda(x)| \leq K$.

Given $g \in H(E)$, we have $g(x)e^{-\Phi(x)} \in L^2(\mathbb{R})$. Therefore, the equality (21) yields $g(w) = \langle g(x), K_w(x) \rangle_{H(E)} = \sum a_\lambda \tilde{K}_\lambda(w)$. Hence, the Bessel property of $\{\zeta_\lambda\}$ implies that

$$\sum_{\lambda \in \Lambda} |a_\lambda|^2 \lesssim \|g(x)e^{-\Phi(x)}\|_{L^2(\mathbb{R})}^2 = \|g\|_{H(E)}^2.$$

This concludes the proof. \square

Following the approach of Beurling [7, pp. 348–358], we treat sampling sequences in \mathcal{F}_Ψ^∞ by looking at certain translations. We need to consider translates, both of the sequence on which we sample and the function being sampled. Since our spaces are not translation invariant, we modify the concept of translations accordingly.

As in Section 3, we consider real valued subharmonic functions Ψ , for which $\mu = \psi'(t)dt$ is a doubling measure on \mathbb{R} induced by $\Delta\Psi$ for some suitable function $\psi \in C^\infty(\mathbb{R})$. Moreover, we suppose that ψ satisfies both (b) and (c) of Lemma 2. I.e., there exist constants such that

$$\psi'(x) \simeq \psi'(y) \text{ whenever } |\psi(x) - \psi(y)| \leq 1. \quad (22)$$

and

$$|\psi''(x)| \lesssim (\psi'(x))^2. \quad (23)$$

Given a point $x_0 \in \mathbb{R}$ we denote by $\tau_{x_0} : \mathbb{C} \rightarrow \mathbb{C}$ the scaled translation

$$\tau_{x_0}(z) = (\psi'(x_0))^{-1}z + x_0.$$

With this, given a set Λ , we make the following definitions,

$$\begin{aligned} \Lambda_{x_0} &= \{\tau_{x_0}^{-1}(\lambda) : \lambda \in \Lambda\}, \\ \psi_{x_0} &= \psi \circ \tau_{x_0} - \psi(x_0), \\ \mu_{x_0}(I) &= \mu(\tau_{x_0}(I)). \end{aligned}$$

Note that evaluating a function f on the sequence Λ is the same as evaluating $f \circ \tau_{x_0}$ on Λ_{x_0} . Also, $\psi_{x_0}(0) = 0$ and $\psi'_{x_0}(0) = 1$, and the Radon-Nikodym derivative of μ_{x_0} is ψ'_{x_0} .

To translate Ψ , we consider the operator

$$K[f](z) = \frac{1}{2\pi} \int_{\mathbb{R}} \left[\log \left| 1 - \frac{z}{t} \right| - \operatorname{Re}(P_q(z/t)) \mathbb{1}_{\mathbb{R} \setminus (-1,1)}(t) \right] f(t) dt,$$

where P_q is the Taylor polynomial of degree q of $\log(1-x)$ around $x=0$. When it exists, the image $K[f]$ is a subharmonic function which satisfies $\Delta K[f] = f(x)dx$. To make the integral converge for $f = \psi'$, observe that since ψ gives a doubling measure, by Remark 2, there exists some $\gamma > 0$ such that $\psi'(x) \lesssim 1 + |x|^\gamma$ for all $x \in \mathbb{R}$. A basic estimate ensures that the choice $q > \gamma + 1$ is sufficient. (The number γ depends only on the doubling constant.) Given $x_0 \in \mathbb{R}$, we now define

$$\Psi_{x_0} := K[\psi'_{x_0}].$$

Recall, that the Fréchet distance between two closed sets $E, F \subseteq \mathbb{R}$ is defined by $[E, F] := \max\{\sup_{x \in E} d(x, F), \sup_{y \in F} d(y, E)\}$. We use Beurling's definition of strong and compactwise limits of sets. A sequence $\{Q_j\}$ of closed sets in \mathbb{R} , converges strongly to a set $Q \subseteq \mathbb{R}$, denoted $Q_j \rightarrow Q$, if $[Q, Q_j] \rightarrow 0$. The sequence Q_j converges compactwise to Q , denoted $Q_j \rightharpoonup Q$, if for every compact set K in \mathbb{R} we have $(K \cap Q_j) \cup \partial K \rightarrow (K \cap Q) \cup \partial K$.

Given a set $\Lambda \subseteq \mathbb{R}$, we say that $\Lambda^* \subseteq \mathbb{R}$ is a weak limit of Λ , if there exists a sequence $\{x_k\} \subseteq \mathbb{R}$ such that $\Lambda_{x_k} \rightharpoonup \Lambda^*$.

Lemma 16. *Let ψ be as above, and suppose that the sequence Λ is ψ -separated. If $\{x_k\} \subseteq \mathbb{R}$ is a sequence, then there exists a subsequence $\{x_{k_n}\}$ such that the following holds.*

- (i) *For some sequence Λ^* , we have $\Lambda_{x_{k_n}} \rightharpoonup \Lambda^*$.*
- (ii) *For a function ψ^* , it holds that $\psi_{x_{k_n}} \rightarrow \psi^*$ and $\psi'_{x_{k_n}} \rightarrow (\psi^*)'$ uniformly on compacts.*
- (iii) *Let ψ^* be as above. Then it gives a doubling measure with doubling constant smaller than the one of $\mu = \psi'(x)dx$. Moreover, $(\psi^*)'$ satisfies (22), with the same constants or better.*
- (iv) *Let ψ^* be as above and set $\Psi^* = K[(\psi^*)']$, then $\Psi_{x_{k_n}} \rightarrow \Psi^*$ uniformly on compacts in \mathbb{C} .*

Remark 6. We write $(\Lambda_{x_{k_n}}, \Psi_{x_{k_n}}, \psi_{x_{k_n}}) \rightharpoonup (\Lambda^*, \Psi^*, \psi^*)$, and denote the set of all such weak limits by $W(\Lambda, \Psi, \psi)$.

Proof. (i): Fix some $a > 0$. For $\tau_{x_k}^{-1}(\lambda) \in \Lambda_{x_k} \cap [-a, a]$, by Remark 3, it holds that $d_\psi(x_k, \lambda) \lesssim a^\alpha$ for some $\alpha > 0$. Since in addition Λ is ψ -separated, we get $\sup_k |(\Lambda_{x_k} \cap [-a, a])| < \infty$. Passing to a subsequence we can assume that the number of points in each such set is constant. By a compactness argument we now find a set Λ_a such that $\Lambda_{x_{k_n}} \cap [-a, a] \rightarrow \Lambda_a$. Letting a tend to infinity, we obtain Λ^* from a usual diagonal argument.

(ii): For t in an interval, there are constants, only depending on the euclidean length, such that $\psi'(\tau_{x_k}(t)) \simeq \psi'(x_k)$. From this it follows that the functions $\psi'_{x_k}(t)$ and $\psi''_{x_k}(t)$ are uniformly bounded on the interval. By the Arzelà-Ascoli theorem and Cantor diagonalization, there exists a subsequence $\{x_{k_n}\}$ and a function ψ^* such that $\psi'_{x_{k_n}} \rightarrow (\psi^*)'$ uniformly on compacts.

(iii): By definition, the measure μ_{x_k} is doubling with a constant not bigger than the constant of μ , and taking limits the statement about $\mu^* = (\psi^*)'(t)dt$ follows. Similarly, to show that (22) holds for ψ^* , it is sufficient to establish it for the ψ_{x_k} .

(iv): To show that $|K[\psi'_{x_k}] - K[(\psi^*)']|$ tends to zero on a given compact, split the domain of integration into a large interval $(-r, r)$ and its complement. The latter part is made small by choosing r and the degree of the Taylor polynomial large enough, depending on the doubling constant of $\psi'(x)dx$. The integral over $(-r, r)$ is then handled by observing that the kernel of the operator is integrable and letting k tend to infinity. \square

In the next lemma, we list the properties preserved by weak limits.

Lemma 17. *Let ψ be as above, and suppose that the sequence Λ is ψ -separated. If $\{x_k\} \subseteq \mathbb{R}$ is a sequence such that $(\psi_{x_k})' \rightarrow (\psi^*)'$ uniformly on compacts for some function ψ^* , and $\Lambda_{x_k} \rightharpoonup \Lambda^*$ for some sequence Λ^* , then the following holds:*

- (i) *Λ^* is ψ^* -separated.*

$$(ii) \ D_{\psi}^{-}(\Lambda) \leq D_{\psi^*}^{-}(\Lambda^*).$$

Proof. (i): First, we note that, if Λ is ψ -separated by some constant, then each Λ_{x_k} is ψ_{x_k} -separated by the same constant. Using the convergence of $(\psi_{x_k})'$ and Λ_{x_k} to $(\psi^*)'$ and Λ^* , respectively, the conclusion follows.

(ii): By the definitions of μ_{x_k} and the density, it follows that for any $\epsilon > 0$ we can choose $r > 0$ sufficiently large to get

$$\frac{\min_{\mu_{x_k}(I)=r} |I \cap \Lambda_{x_k}|}{r} = \frac{\min_{\mu(J)=r} |J \cap \Lambda|}{r} > D_{\psi}^{-}(\Lambda) - \epsilon.$$

Observe that for big enough k , we have $|(I \cap \Lambda^*)| - |(I \cap \Lambda_{x_k})| \leq 2$ (this is because Fréchet limits allows an extra element on each endpoint of I , and the sequences are uniformly separated). In addition, if $\mu^*(I) = r$ then $\mu_{x_k}(I) > (1 - \epsilon)r$ since $\psi_{x_k} \rightarrow \psi^*$ uniformly on compacts, and so we obtain for big enough r that

$$\frac{\min_{\mu^*(I)=r} |I \cap \Lambda^*| + 2}{r} \geq (1 - \epsilon)(D_{\psi}^{-}(\Lambda) - \epsilon).$$

By taking limits we get the result. \square

Given $x_0 \in \mathbb{R}$, let H_{x_0} be an entire function with $\Psi_{x_0} - \Psi \circ \tau_{x_0}$ as its real part, and define an operator $T_{x_0} : \mathcal{F}_{\Psi}^{\infty} \rightarrow \mathcal{F}_{\Psi_{x_0}}^{\infty}$ by

$$T_{x_0} : f \mapsto f \circ \tau_{x_0} e^{H_{x_0}}.$$

The proof of the following lemma is straight-forward and omitted.

Lemma 18. *Let Ψ and ψ be as above.*

(i) T_{x_0} is an isometry between $\mathcal{F}_{\Psi}^{\infty}$ and $\mathcal{F}_{\Psi_{x_0}}^{\infty}$. Moreover

$$|T_{x_0} f(z)| e^{-\Psi_{x_0}(z)} = |f \circ \tau_{x_0}(z)| e^{-\Psi \circ \tau_{x_0}(z)}. \quad (24)$$

(ii) Suppose that $\{f_k\}$ is a sequence of functions in $\mathcal{F}_{\Psi}^{\infty}$, uniformly bounded in norm, and let $\{x_k\} \subseteq \mathbb{R}$. If $\psi'_{x_k} \rightarrow (\psi^*)'$ uniformly on compacts, then there exists a subsequence $\{x_{k_n}\}$ for which $T_{x_{k_n}} f_{k_n}$ converges uniformly on compacts to a function in $\mathcal{F}_{\Psi^*}^{\infty}$.

It is not difficult to see that if Λ is a ψ -separated sampling sequence for $\mathcal{F}_{\Psi}^{\infty}$ and $(\Lambda^*, \Psi^*, \psi^*) \in W(\Lambda, \Psi, \psi)$, then Λ^* is a sampling sequence for $\mathcal{F}_{\Psi^*}^{\infty}$ and, in particular, a uniqueness set for the space. We will need the opposite direction of this claim, which is given in the following lemma.

Lemma 19. *Let Ψ and ψ be as above, and suppose that the sequence Λ is ψ -separated. If the sequence Λ^* is a uniqueness set for $\mathcal{F}_{\Psi^*}^{\infty}$ whenever $(\Lambda^*, \Psi^*, \psi^*) \in W(\Lambda, \Psi, \psi)$, then Λ is a sampling sequence for $\mathcal{F}_{\Psi}^{\infty}$.*

Proof. Assume that Λ is not a sampling sequence for $\mathcal{F}_{\Psi}^{\infty}$. This means that for every $k \in \mathbb{N}$, there exists a function f_k of unit norm and $x_k \in \mathbb{R}$ such that

$$\sup_{\lambda \in \Lambda} |f_k(\lambda)| e^{-\Psi(\lambda)} \leq \frac{1}{k} \quad \text{and} \quad |f_k(x_k)| e^{-\Psi(x_k)} > 1/2.$$

By lemma 16, there exists a subsequence $\{x_{k_n}\}$ such that $(\Lambda_{x_{k_n}}, \Psi_{x_{k_n}}, \psi_{x_{k_n}}) \rightharpoonup (\Lambda^*, \Psi^*, \psi^*)$. By Lemma 18 (taking another subsequence if needed), $T_{x_{k_n}} f_{k_n}$ converges uniformly on compacts to a function $f^* \in \mathcal{F}_{\Psi^*}^\infty$. We claim that on the one hand, $f^* \not\equiv 0$ and on the other hand, $f^*|_{\Lambda^*} = 0$. This contradicts our assumption and therefore completes the proof. We omit the details. \square

We can now use the previous lemma to connect between the density of a sequence and its sampling properties.

Lemma 20. *Let Ψ and ψ be as above, and suppose that the sequence Λ is ψ -separated. If $D_\psi^-(\Lambda) > 1/2\pi$, then Λ is a sampling sequence for \mathcal{F}_Ψ^∞ .*

Proof. By Lemma 19, it is enough to show that if $(\Lambda^*, \Psi^*, \psi^*) \in W(\Lambda, \Psi, \psi)$, and $f \in \mathcal{F}_{\Psi^*}^\infty$ satisfies $f|_{\Lambda^*} = 0$, then $f \equiv 0$. Assume to the contrary that there exists such f which is non-zero. We may assume, without loss of generality, that $f(0) \neq 0$ (otherwise we translate the setting as above). Choose $r > 0$, by Jensen's formula applied to f , we get

$$\int_0^r \frac{|\Lambda^* \cap [-t, t]|}{t} dt \leq \frac{1}{2\pi} \int_0^{2\pi} \Psi^*(re^{i\theta}) d\theta + O(1).$$

Since $\Delta(\Psi^*(rz)) = r(\psi^*)'(rx)dx$, Greens formula yields that the integral on the right-hand side of the last expression is equal to

$$\begin{aligned} \frac{1}{2\pi} r \int_{-1}^1 \log \frac{1}{|x|} \cdot (\psi^*)'(rx) dx &= \frac{1}{2\pi} \int_{1 \leq |t| \leq r} \frac{\psi^*(t)}{|t|} dt + O(1) \\ &= \frac{1}{2\pi} \int_1^r \frac{\mu^*([-t, t])}{t} dt + O(1). \end{aligned}$$

This contradicts our assumption about the density of Λ , which, by Lemma 17, implies that there exists some $\epsilon > 0$, so that for any t big enough,

$$\frac{|\Lambda^* \cap [-t, t]|}{\mu^*([-t, t])} \geq \frac{1 + \epsilon}{2\pi}.$$

This ends the proof. \square

We are now ready to prove the main result.

Proof of Theorem 1. Let Φ be defined by (2), and recall that $\Delta\Phi = 2\phi'(x)dx$. Since $D_{2\phi}^-(\Lambda) > \frac{1}{2\pi}$, there exists an $\epsilon > 0$ small enough for $D_{(1+\epsilon)2\phi}^-(\Lambda) > \frac{1}{2\pi}$ to hold. Set $\Psi = (1+\epsilon)\Phi$, and observe that $\Delta\Psi = (1+\epsilon)2\phi'(x)dx$. So, by Lemma 20, the sequence Λ is sampling for \mathcal{F}_Ψ^∞ . The result now follows from Lemma 15. \square

ACKNOWLEDGEMENT

The authors would like to thank Joaquim Ortega-Cerdà, Kristian Seip, Alexandru Aleman and Anton Baranov for some helpful conversations.

REFERENCES

1. A. B. Aleksandrov, *A simple proof of the Vol'berg-Treil' theorem on the embedding of co-variant subspaces of the shift operator*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **217** (1994), no. Issled. po Linein. Oper. i Teor. Funktsii. 22, 26–35, 218.
2. ———, *Embedding theorems for coinvariant subspaces of the shift operator. II*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **262** (1999), no. Issled. po Linein. Oper. i Teor. Funkts. 27, 5–48, 231.
3. Anton D. Baranov, *Bernstein-type inequalities for shift-coinvariant subspaces and their applications to Carleson embeddings*, J. Funct. Anal. **223** (2005), no. 1, 116–146.
4. ———, *Stability of bases and frames of reproducing kernels in model spaces*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 7, 2399–2422.
5. Anton D. Baranov and Konstantin Dyakonov, *The Feichtinger conjecture for reproducing kernels in model subspaces*, available on www.arxiv.org, 2009.
6. Yurii Belov, Tesfa Mengestie, and Kristian Seip, *Discrete hilbert transforms on sparse sequences*, arXiv:0912.2899v1, 2009.
7. Arne Beurling, *The collected works of Arne Beurling. Vol. 2*, Contemporary Mathematicians, Birkhäuser Boston Inc., Boston, MA, 1989, Harmonic analysis, Edited by L. Carleson, P. Malliavin, J. Neuberger and J. Wermer.
8. Alexander Borichev, Remi Duhuez, and Karim Kellay, *Sampling and interpolation in large Bergman and Fock spaces*, J. Funct. Anal. **242** (2007), no. 2, 563–606.
9. Alexander Borichev and Yurii Lyubarskii, *Riesz bases of reproducing kernels in Fock-type spaces*, J. Inst. Math. Jussieu **9** (2010), no. 3, 449–461.
10. Lennart Carleson, *Interpolations by bounded analytic functions and the Corona problem*, Ann. of Math. **76** (1962), no. 3, 547–559.
11. Bill Cohn, *Carleson measures for functions orthogonal to invariant subspaces*, Pacific J. Math. **103** (1982), no. 2, 347–364.
12. William S. Cohn, *Carleson measures and operators on star-invariant subspaces*, J. Operator Theory **15** (1986), no. 1, 181–202.
13. Louis de Branges, *Hilbert spaces of entire functions*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1968.
14. Peter Duren and Alexander Schuster, *Bergman spaces*, Mathematical Surveys and Monographs, vol. 100, American Mathematical Society, Providence, RI, 2004.
15. Victor Havin and Javad Mashreghi, *Admissible majorants for model subspaces of H^2 . I. Slow winding of the generating inner function*, Canad. J. Math. **55** (2003), no. 6, 1231–1263.
16. S. Jaffard, *A density criterion for frames of complex exponentials*, Michigan Math. J. **38** (1991), no. 3, 339–348.
17. Yurii I. Lyubarskii and Kristian Seip, *Weighted Paley-Wiener spaces*, J. Amer. Math. Soc. **15** (2002), no. 4, 979–1006 (electronic).
18. I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications.
19. Nicolas Marco, Xavier Massaneda, and Joaquim Ortega-Cerdà, *Interpolating and sampling sequences for entire functions*, Geom. Funct. Anal. **13** (2003), no. 4, 862–914.
20. Giuseppe Mastroianni and Vilmos Totik, *Weighted polynomial inequalities with doubling and A_∞ weights*, Constr. Approx. **16** (2000), no. 1, 37–71.
21. Tesfa Y. Mengestie, Doctoral dissertation, Norwegian University of Science and Technology, in preparation.
22. Joaquim Ortega-Cerdà, *Multipliers and weighted $\bar{\partial}$ -estimates*, Rev. Mat. Iberoamericana **18** (2002), no. 2, 355–377.
23. ———, private communication, 2010.

24. Joaquim Ortega-Cerdà and Kristian Seip, *Beurling-type density theorems for weighted L^p spaces of entire functions*, J. Anal. Math. **75** (1998), 247–266.
25. Jayakumar Ramanathan and Tim Steger, *Incompleteness of sparse coherent states*, Appl. Comput. Harmon. Anal. **2** (1995), no. 2, 148–153.
26. Kristian Seip, *Density theorems for sampling and interpolation in the Bargmann-Fock space. I*, J. Reine Angew. Math. **429** (1992), 91–106.
27. ———, *On the connection between exponential bases and certain related sequences in $L^2(-\pi, \pi)$* , J. Funct. Anal. **130** (1995), no. 1, 131–160.
28. Kristian Seip, *Interpolation and Sampling in Spaces of Analytic Functions*, University Lecture Series, vol. 33, American Mathematical Society, Providence, R. I., 2004.
29. Kristian Seip and Robert Wallstén, *Density theorems for sampling and interpolation in the Bargmann-Fock space. II*, J. Reine Angew. Math. **429** (1992), 107–113.
30. Harold S. Shapiro and Allen L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math **83** (1961), 513–532.
31. A. L. Vol'berg and S. R. Treil', *Embedding theorems for invariant subspaces of the inverse shift operator*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **149** (1986), no. Issled. Linein. Teor. Funktsii. XV, 38–51, 186–187.
32. Robert M. Young, *An Introduction to Nonharmonic Fourier Series*, first ed., Academic Press Inc., New York, NY, 1980.

DEPARTAMENT DE MATEMÀTICA APLICADA I ANÀLISI, UNIVERSITAT DE BARCELONA, GRAN VIA 585, 08007, BARCELONA, SPAIN

E-mail address: jmarzo@ub.edu

©SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY RAMAT AVIV, TEL AVIV 69978, ISRAEL

E-mail address: shahaf.n.h@gmail.com

CENTRE FOR MATHEMATICAL SCIENCES, LUND UNIVERSITY, P.O. Box 118, SE-221 00 LUND, SWEDEN

E-mail address: janfreol@maths.lth.se